Walking and Running
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Walking and running

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Complex bodies

The human body is a complicated machine whose movements involve many different joints, operated by a great many muscles. For that reason it is easy to get bogged down in detail when thinking about walking and running from a mathematical point of view.

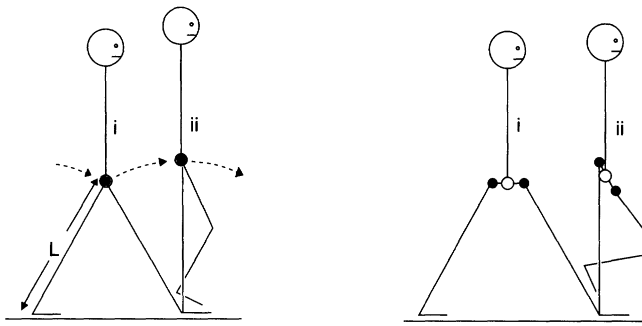
Any position of the human body (or of any other jointed mechanism) can be described by giving the angles of joints. The number of angles needed for an unambiguous description is the number of *degrees of freedom* of the mechanism. For example, the position of a hinge joint is described by just one angle: a hinge allows only one degree of freedom. The human knee is a hinge. The ankle, however, allows rotation about two axes – you can tilt your foot toes up or toes down, and you can also rock it sideways so that the sole faces inwards towards the other foot – so it gives two degrees of freedom. The hip is a ball and socket joint allowing rotation about any axis through the centre of the ball, but any position can be described by just three angles (measured, for example, in three planes at right angles to each other), so it allows three degrees of freedom. In total, there are six degrees of freedom in each leg, making twelve in all, and suggesting that we need twelve equations of motion to describe walking. If we took account of the flexibility of the foot and the movements of the arms, we would need more.

A set of twelve simultaneous equations is daunting enough, but the problem seems worse when we look more deeply into the anatomy of the leg. One mathematical model of walking identified 29 important muscles in each leg, 58 in all. You may want to know the force each muscle should exert, at each stage of the stride, but you cannot evaluate 58 unknown variables by solving a set of only twelve simultaneous equations.

A simple model

The situation looks less bad when you devise simplified models. Look at someone walking and try to find the essence of the movement. The feet move alternately, each being lifted just after the other is set down. While a foot is on the ground, that leg's knee is almost straight, keeping the distance from hip to ankle almost constant. As a result, our bodies rise and fall (by about three centimetres) in each step. We are highest when the straight, supporting leg is vertical.

These features of walking are represented in the ultra-simple model shown in Figure 1(a). Each foot is set down as the other is lifted. While its foot is on the ground, each leg is straight, so the body rises and falls in a series of arcs of circles. We will ignore the masses of the legs, putting all the mass in the rigid trunk.



(a) A simple model of human walking

(b) A diagram of the racing walk. By tilting the hips at stage (ii), the athlete avoids raising the body's centre of mass as high as she or he otherwise would.

FIGURE 1

Now we have to remember that a body moving in an arc of a circle has an acceleration towards the centre of the circle. This acceleration is (speed^2)/radius: in the case of our walker, the radius is leg length L, so if the speed is v the acceleration is v^2/L. At stage (ii) (Figure 1(a)) this acceleration is vertically downwards. The walker cannot fall with an acceleration greater than gravitational g so

$$v^2/L \leqslant g$$

$$v \leqslant (gL)^{1/2} \qquad (1)$$

This equation tells us that there is a maximum possible walking speed. My legs are 0.9 metre long and the gravitational acceleration is 10 m/s^2 so my theoretical maximum walking speed is $(10 \times 0.9)^{1/2} = 3$ m/s. The fastest I can actually walk without breaking into a run is a little less than that, about 2.6 m/s.

Human adults (including me) prefer to change from walking to running when our speed reaches 2 m/s. Measurements of oxygen consumption show that below that speed walking needs less energy than running; and above 2 m/s, running is the more economical gait. But we can walk at speeds approaching 3 m/s. And (with an exception to be mentioned soon) we cannot walk faster.

My grandson, aged two, has much shorter legs than mine, only 0.4 m long. The theory says he cannot walk faster than $(10 \times 0.4)^{1/2} = 2$ m/s. That seems to explain why, unless I walk slowly, he has to run to keep up with me.

The gravitational acceleration is 10 m/s^2 here but only 1.6 m/s^2 on the moon. Maximum walking speed there should be $(1.6 \times 0.9)^{1/2} = 1.2$ m/s. The astronauts did not walk on the moon but bounded, perhaps because walking would have been intolerably slow.

So far, the simple theory is doing well. It has explained why adults cannot walk faster than 3 m/s, why small children have to run to keep up with us and why the astronauts did not walk on the moon. But here is a puzzle. Athletes in walking races travel at up to 4.4 m/s (that is the men's world record speed, for the 10 kilometre walk). Can champion athletes break the laws of mechanics? Of course they cannot. They go faster than the model could go by moving in a way the model cannot, wiggling their hips so as to reduce the rise and fall of the body's centre of mass (Figure 1(b)). That reduces the vertical acceleration needed, for travelling at any given speed. The model can only tell us the maximum speed for a walker whose movements match the model's assumptions.

Dynamic similarity

We humans use just two gaits, walking and running. Horses, dogs and other four-legged mammals have three principal gaits: the walk, trot and gallop. Can we predict the speeds at which different-sized mammals should change gaits? If a horse breaks into a gallop at 6 m/s, at what speed should a dog do the same?

Non-human animals do not walk on straight legs, so Figure 1(a) does not model them well. We will try a more general approach, starting from the familiar concept of geometric similarity. Two shapes are geometrically similar if one is a scale model of the other: that is, if one could be made identical to the other by multiplying all its linear dimensions by the same factor. Animals of different sizes are not geometrically similar (a horse is not just an oversized dog) but there is enough similarity for it to seem sensible to ask, what if they *were* geometrically similar?

Dynamic similarity is a concept that applies to moving systems. Two systems are dynamically similar if one could be made identical to the other by multiplying all lengths by one factor; all times by another factor; and all forces by a third factor. For example, two pendulums of different lengths, swinging through the same angle, have dynamically similar motion. In what circumstances could different-sized animals move in dynamically similar ways? Their bodies must be geometrically similar, and they must move at appropriate speeds.

As animals walk or run, they rise and fall in each stride, gaining and losing potential energy. They also speed up and slow down in each stride, gaining and losing kinetic energy. Some energy gets swapped back and forth (pendulum fashion) between the potential and kinetic forms. It seems clear (and it can be shown more rigorously) that dynamic similarity will be possible only if the two animals have equal ratios of potential to kinetic energy. If an animal of mass m is supported on legs of length L it has potential energy mgL. If it is travelling at speed v, its kinetic energy is $\frac{1}{2}mv^2$.

$$\frac{\text{kinetic energy}}{\text{potential energy}} = \frac{\frac{1}{2}mv^2}{mgL} = \frac{1}{2}\frac{v^2}{gL}. \tag{2}$$

Dynamic similarity is possible for animals of different sizes only if their speeds are such as to give them equal values of (v^2/gL).

The dimensionless quantity v^2/gL is called a *Froude* number after the Victorian engineer who used similar numbers in an analysis of wave resistance to the motion of ships – but that is another story. It links with our earlier discussion because equation (1) can be written in a different way:

$$\text{for walking} \qquad v^2/gL \leqslant 1. \tag{3}$$

You cannot walk with a Froude number greater than one (unless, of course, you use the racing walk).

The advantage of the Froude number concept is that it is not tied to a particular model of walking, such as the one shown in Figure 1(a). It suggests a general hypothesis: similar-shaped animals of different sizes will tend to move in dynamically similar ways whenever possible – that is, when their Froude numbers are equal. They will make corresponding gait changes (from walk to trot, or from trot to gallop) at speeds which make their Froude numbers equal. Compare a dog with a horse whose legs are four times as long. To have the same Froude number, the horse must travel at twice the speed of the dog (two is the square root of four). It should make each gait change at twice the speed at which the dog makes the same change. This turns out to be roughly true. More generally, mammals ranging from small rodents to giraffes change from walking to trotting at a Froude number of about 0.5, and from trotting to galloping at a Froude number of about 2.5.

Dynamically similar movement implies more than just changing gaits at equal Froude numbers. It implies, for example, taking strides in proportion to leg length: when dogs and horses travel with equal Froude numbers we can expect the horses' strides to be four times as long as the dogs'. More generally, we can expect graphs of (stride length/leg length) against Froude number to be the same for different-sized animals. Figure 2 shows that this is reasonably nearly true for animals ranging from dogs to elephants, and including two-legged as well as four-legged species. Do not be surprised that the same relationship holds for bipeds as for quadrupeds: except when it gallops, a horse resembles two people running one behind the other.

Dinosaur speeds

Figure 2 has been used in an unexpected way, to estimate the speeds of dinosaurs. Footprints have been found of many kinds of dinosaurs – footprints made in mud which has subsequently changed to stone. From these we can measure stride length, the distance from (say) a left hind footprint to the next print of the same foot. Also, from the size of the prints, we can estimate the size of the dinosaur that made them. Thus we can estimate (stride length/leg length). With this information, using the graph, we can estimate the Froude number at which the dinosaurs were walking or running – and since we have an estimate of leg length, we can translate the Froude number to a speed. This approach enables us to calculate, very roughly, the speeds of dinosaurs from the spacing of their footprints.

The results may seem just a little disappointing: big dinosaurs usually moved slowly. Quite a lot of footprints have been found in Texas and elsewhere of large brontosaurs, animals weighing around 30 tonnes, or six

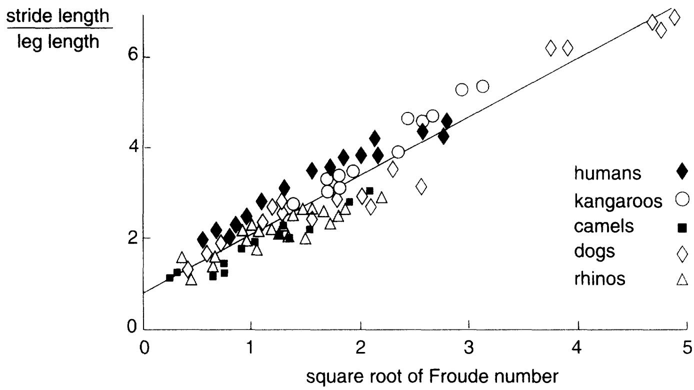


FIGURE 2 A graph of (stride length/leg length) against the square root of Froude number for humans and various animals. Each animal is represented by several points, recording observations of it moving at different speeds. The square root of Froude number is plotted, rather than Froude number itself, to avoid undue clumping of points in the lower range of speeds.

times as much as a large elephant. All these seem to show speeds of about one metre per second, which is a slow walking speed for humans and seems painfully slow for such giants. Footprints have also been found of carnivorous (tyrannosaur-like) dinosaurs of up to 5 tonnes. These show speeds of around 2 m/s, a fast walk for humans. Large dinosaurs seem to have moved slowly, at human walking speeds. They may occasionally have gone faster, but no footprints of their running have been found. There are some footprints of smaller dinosaurs going faster; notably a half-tonne biped going at an estimated 12 m/s, which is faster than any human athlete (up to 11 m/s) but considerably slower than a racehorse (17 m/s). Regrettably we have no way of checking whether these estimates of dinosaur speeds are correct.

Some biomechanicists have devised detailed mechanical models of the human body, imitating as much as possible of its complexity. Some of these models have given valuable insight into movements such as walking and running. But this article has shown that a very simple model (Figure 1(a)) can illuminate the basic principles of walking and running. And the simple concept of dynamic similarity has enabled us to draw together information about the movements of a wide range of living animals. It has even enabled us to speculate about dinosaur speeds.

Further reading

R. McN. Alexander, *Dynamics of dinosaurs and other extinct giants*, Columbia University Press, New York (1989).

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